

A renormalized Riemann-Roch formula and the Thom isomorphism for the free loop space

Matthew Ando and Jack Morava

ABSTRACT. Let E be a circle-equivariant complex-orientable cohomology theory. We show that the fixed-point formula applied to the free loop space of a manifold X can be understood as a Riemann-Roch formula for the quotient of the formal group of E by a free cyclic subgroup. The quotient is not representable, but (locally at p) its p -torsion subgroup is, by a p -divisible group of height one greater than the formal group of E .

I believe in the fundamental interconnectedness of all things.

—Dirk Gently [Ada88]

1. Introduction

Let \mathbb{T} denote the circle group, and, if X is a compact smooth manifold, let $\mathcal{L}X \stackrel{\text{def}}{=} C^\infty(\mathbb{T}, X)$ denote its free loop space. The group \mathbb{T} acts on $\mathcal{L}X$, and the fixed point manifold is again X , considered as the subspace of constant loops. In the 1980's, Witten showed that the fixed-point formula in ordinary equivariant cohomology, applied to the free loop space $\mathcal{L}X$ of a spin manifold X , yields the index of the Dirac operator (i.e. the \hat{A} -genus) of X —a fundamentally K -theoretic quantity [Ati85]. He also applied the fixed-point theorem in equivariant K -theory to a Dirac-like operator on $\mathcal{L}X$ to obtain the elliptic genus and “Witten genus” of X [Wit88]—quantities associated with elliptic cohomology.

Among homotopy theorists, these developments generated considerable excitement. The chromatic program organizes the structure of finite stable homotopy types, locally at a prime p , into layers indexed by nonnegative integers. The n th layer is detected by a family of cohomology theories \mathcal{E}_n ; rational cohomology, K -theory, and elliptic cohomology are detecting theories for the first three layers [Mor85, DHS88, HS98].

The geometry and analysis related to rational cohomology and K -theory are reasonably well-understood, but for $n \geq 2$ and for elliptic cohomology in particular, very little is known. Witten's work provides a major suggestion: for $n = 1$ and $n = 2$ his analysis gives a correspondence

$$\begin{array}{ccc} \text{analysis underlying } \mathcal{E}_n & \leftrightarrow & \text{analysis underlying } \mathcal{E}_{n-1} \\ \text{applied to } X & & \text{applied to } \mathcal{L}X. \end{array} \quad (1.0.1)$$

This paper represents our attempt to understand why Witten's procedure appears to connect the chromatic layers in the manner of (1.0.1). To do this we consider very generally the fixed-point formula attached to a complex-oriented theory E with formal group law F . We recall that for $n > 0$, such a theory detects chromatic layer n if the formal group law F has height n .

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Our first result is that the fixed-point formula of a suitable equivariant extension of E (Borel cohomology is fine, as is the usual equivariant K -theory) applied to the free loop space yields a formula which is *identical* to the Riemann-Roch formula for the quotient $F/(\hat{q})$ of the formal group law F by a free cyclic subgroup (\hat{q}) (compare formulae (3.2.2) and (4.2.3)).

The quotient $F/(\hat{q})$ is not a formal group, so to understand its structure, we work p -locally and study its p -torsion subgroup $F/(\hat{q})[p^\infty]$. We construct a group $\text{Tate}(F)$ with a canonical map

$$\text{Tate}(F) \rightarrow F/(\hat{q}),$$

which induces an isomorphism of torsion subgroups in a suitable setting. Our second result is that the group $\text{Tate}(F)[p^\infty]$ is a p -divisible group, fitting into an extension

$$F[p^\infty] \rightarrow \text{Tate}(F) \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$$

of p -divisible groups. If the height of F is n , then the height of $\text{Tate}(F)[p^\infty]$ is $n+1$, but its étale quotient has height 1. In a sense we make precise in §5.3, it is the universal such extension.

Thus the fixed-point formula on the free loop space interpolates between the chromatic layers in the same way that p -divisible groups of height $n+1$ with étale quotient of height 1 interpolate between formal groups of height n and formal groups of height $n+1$. This is discussed in more detail, from the homotopy-theoretic point of view, in our earlier paper [AMS98] with Hal Sadofsky; this paper is a kind of continuation, concerned with analytic aspects of these phenomena. We show that Witten's construction in rational cohomology produces K -theoretic genera because of the exponential exact sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C} \rightarrow \mathbb{C}^\times \rightarrow 1 \quad (1.0.2)$$

expressing the multiplicative group (K -theory) as the quotient of the additive group (ordinary cohomology) by a free cyclic subgroup; while his work in K -theory produces elliptic genera because of the exact sequence

$$0 \rightarrow q^\mathbb{Z} \rightarrow \mathbb{C}^\times \rightarrow \mathbb{C}^\times/q^\mathbb{Z} \rightarrow 1 \quad (1.0.3)$$

(where q is a complex number with $|q| < 1$), expressing the Tate elliptic curve $\mathbb{C}^\times/q^\mathbb{Z}$ as the quotient of the multiplicative group by a free cyclic subgroup.

These analytic quotients have already been put to good use in equivariant topology. Grojnowski constructs from equivariant ordinary cohomology a complex \mathbb{T} -equivariant elliptic cohomology using the elliptic curve \mathbb{C}/Λ which is the quotient of the complex plane by a lattice; and Rosu uses Grojnowski's functor to give a striking conceptual proof of the rigidity of the elliptic genus. Grojnowski's ideas applied to the multiplicative sequence (1.0.3) give a construction of complex \mathbb{T} -equivariant elliptic cohomology based on equivariant K -theory; details will appear elsewhere. Completing this circle, Rosu has used the quotient (1.0.2) to give a construction of complex equivariant K -theory [Gro94, Ros99, RK99].

Several of the formulae in this paper involve formal infinite products; see for example (3.2.2) and (4.2.3). On the fixed-point formula side, the source of these is the Euler class of the normal bundle ν of X in $\mathcal{L}X$ (3.1.2). From this point of view, the problem is that the bundle ν is infinite-dimensional, so it does not have a Thom spectrum in the usual sense. However, ν has a highly nontrivial circle action, which defines a locally finite-dimensional filtration by eigenspaces. Following the program sketched in [CJS95], we construct from this filtration a Thom *pro-spectrum*, whose Thom class is the infinite product.

In the particular cases of the additive and multiplicative formal groups ($n = 1, 2$ above), one can also control the infinite products by replacing them with products which converge to holomorphic functions on \mathbb{C} ; this construction of elliptic functions goes back to Eisenstein. We are grateful to Kapranov for pointing out to us that Eisenstein considered the the analogous problem for $n > 2$. In [Eis44] he described the difficulty of interpreting such infinite products. He went on to hint that he perceived a useful approach, and concluded the following.

Die Functionen, zu welchen man auf diesen Wege geführt wird, scheinen sehr merkwürdige Eigenschaften zu besitzen; sie eröffnen ein Feld, auf dem sich Stoff zu den reichhaltigsten Untersuchungen darbietet, und welches der eigentliche Grund und Boden zu sein scheint, auf welchem die schwierigsten Theile der Analysis und Zahlentheorie ineinander greifen.

1.1. Formal group schemes. In this paper (especially in section 5) we shall consider formal schemes in the sense of [Str99, Dem72]. A *formal scheme* is a filtered colimit of affine schemes. For example the “formal line”

$$\hat{\mathbb{A}}^1 \stackrel{\text{def}}{=} \operatorname{colim}_n \operatorname{spec} \mathbb{Z}[x]/x^n$$

is a formal scheme. Note that an affine scheme is a formal scheme in a trivial way. An important feature of this category which we shall use is that it has finite products. For example,

$$\hat{\mathbb{A}}^1 \times \hat{\mathbb{A}}^1 = \operatorname{colim} \operatorname{spec} (\mathbb{Z}[x]/(x^n) \otimes \mathbb{Z}[y]/(y^m)).$$

In particular a *formal group scheme* means an abelian group in the category of formal schemes. A formal group scheme whose underlying formal scheme is isomorphic to the formal scheme $\hat{\mathbb{A}}^1$ is called a commutative one-dimensional formal Lie group. We shall simply call it a *formal group*.

The first reason for considering formal schemes is that formal groups are not quite groups in the category of affine schemes, because a group law

$$F(s, t) = s + t + \cdots \in R[[s, t]]$$

over a ring R gives a diagonal

$$R[[s]] \rightarrow R[[s, t]] \cong R[[s]] \hat{\otimes} R[[t]]$$

only to the completed tensor product.

The second reason for considering formal schemes is that, if G is an affine group scheme, then its torsion subgroup G_{tors} is a formal scheme (the colimit of the affine schemes $G[N]$ of torsion of order N), but not in general a scheme.

If X is a formal scheme over R , and S is an R -algebra, then X_S will denote the resulting formal scheme over S .

2. The umkehr homomorphism and an ungraded analogue

2.1. Let E be a complex-oriented multiplicative cohomology theory with formal group law F , and let $h: X \rightarrow Y$ be a proper complex-oriented map of smooth finite-dimensional connected manifolds, of fiber dimension $d = \dim X - \dim Y$. The Pontrjagin-Thom collapse associates to these data an “umkehr” homomorphism [Qui71]

$$h_! : E^*(X) \rightarrow E^{*-d}(Y).$$

We will be concerned with similar homomorphisms in certain infinite-dimensional contexts. In order to do so, we systematically eliminate the shift of $-d$ in the degree by restricting our attention to *even periodic* cohomology theories E . The examples show (3.3) that this amounts to measuring quantities relative to the vacuum.

2.2. Even periodic ring theories. Let E be a cohomology theory. If X is a space, then $E^*(X)$ will denote its *unreduced* cohomology; if A is a spectrum, then $E^*(A)$ will denote its cohomology in the usual sense. These notations are related by the isomorphism $E^*(X) \cong E^*(\Sigma^\infty X_+)$, where X_+ denotes the union of X and a disjoint basepoint. The reduced cohomology of X will be denoted $\tilde{E}(X)$. Let $*$ denote the one-point space.

A cohomology theory E with commutative multiplication is *even* if $E^{\text{odd}}(*) = 0$. It is *periodic* if $E^2(*)$ contains a unit of $E^*(*)$. If E is an even periodic theory, then we write $E(X)$ for $E^0(X)$ and E for $E^0(*)$. We sometimes write $X_E = \operatorname{spec} E(X)$ for the spectrum, in the sense of commutative algebra, of the commutative ring $E(X)$.

A space X is *even* if $H_*(X)$ is a free abelian group, concentrated in even degrees. In that case the natural map

$$\operatorname{colim} F_E \rightarrow X_E, \tag{2.2.1}$$

where F is the filtered system of maps of finite CW complexes to X , is an isomorphism. This gives X_E the structure of a formal scheme. The functor $X \mapsto X_E$ from even spaces to formal schemes over E preserves finite products and coproducts: if X and Y are two even spaces, then

$$(X \times Y)_E \cong X_E \times Y_E \cong \operatorname{spec} E(X) \hat{\otimes} E(Y).$$

Here $\hat{\otimes}$ refers to the completion of the tensor product with respect to the topology defined by the filtrations of $E(X)$ and $E(Y)$.

2.3. Orientations and coordinates. Let $P \stackrel{\text{def}}{=} \mathbb{C}P^\infty$ be the classifying space for complex line bundles. Let $m: P \times P \rightarrow P$ be the map classifying the tensor product of line bundles. It induces a map

$$P_E \times P_E \xrightarrow{m_E} P_E,$$

which makes P_E a formal group scheme over E . Of course it is a formal group: let $i: S^2 \rightarrow P$ be the map classifying the Hopf bundle. A choice of element $x \in \tilde{E}(P)$ such that $v = i^*x \in \tilde{E}(S^2) \cong E^{-2}(*)$ is a unit is called a *coordinate* on P_E . There is then an isomorphism

$$E(P) \cong E[[x]],$$

which determines a formal group law F over E by the formula

$$F(x, y) = m^*x \in E(P \times P) \cong E[[x, y]].$$

Any even-periodic cohomology theory E is complex-orientable. An *orientation* on E is a multiplicative natural transformation

$$MU \rightarrow E.$$

These correspond bijectively with elements $u \in \tilde{E}^2(P)$ such that

$$i^*u = \Sigma^2(1), \tag{2.3.1}$$

where Σ is the suspension isomorphism [Ada74]. A coordinate x thus determines an orientation $u = v^{-1}x$.

DEFINITION 2.3.2. We shall use the notation (E, x, F) to denote an even periodic cohomology theory E with coordinate x and group law F . We shall call such a triple a *parametrized theory*.

2.4. Thom isomorphism. An orientation $u \in \tilde{E}^2(P)$ gives the usual Thom classes and characteristic classes for complex vector bundles. If k is an integer, let \underline{k} denote the trivial complex vector bundle of rank k . If X is a connected space and V is a complex vector bundle of rank d over X , then we write

$$X^V \stackrel{\text{def}}{=} \Sigma^\infty(\mathbb{P}(V \oplus \underline{1})/\mathbb{P}(V))$$

for the suspension spectrum of its Thom space, with bottom cell in degree $2d$. We write α_{usual}^V for the Thom isomorphism

$$\alpha_{\text{usual}}^V: E^*(X) \cong E^{*+2d}(X^V).$$

In the same way, a coordinate $x \in \tilde{E}(P)$ gives rise to a Thom isomorphism

$$\alpha^V: E(X) \cong E(X^V).$$

If $v = i^*x \in \tilde{E}(S^2)$ is the associated orientation, the isomorphisms α_{usual} and α are related by the formula

$$\alpha^V = v^{\operatorname{rank} V} \alpha_{\text{usual}}^V.$$

REMARK 2.4.1. One effect of condition (2.3.1) is that α_{usual}^d coincides with the suspension isomorphism

$$\alpha_{\text{usual}}^d = \Sigma^{2d}: E^*(X) \cong E^{*+2d}(X^d).$$

The Thom isomorphism α defined by a coordinate chooses $v \in \tilde{E}(S^2) \cong E(*^1)$ as α^1 . Thus α_{usual} may be viewed as a composition of Thom isomorphisms

$$\alpha_{\text{usual}}^V: E(X^d) \xrightarrow{(\alpha^d)^{-1}} E(X) \xrightarrow{\alpha^V} E(X^V).$$

If $\zeta: \Sigma^\infty X_+ \rightarrow X^V$ denotes the zero section, then we write

$$\begin{aligned} e_{\text{usual}}(V) &\stackrel{\text{def}}{=} \zeta^* \alpha_{\text{usual}}^V(1) \in E^{2d}(X) \\ e(V) &\stackrel{\text{def}}{=} \zeta^* \alpha^V(1) \in E(X) \end{aligned}$$

for the usual and degree-zero Euler classes of V ; these are related by the formula

$$e(V) = v^{\text{rank } V} e_{\text{usual}}(V).$$

If $U(n)$ denotes the unitary group and T is its maximal torus of diagonal matrices, then the map

$$E(BU(n)) \rightarrow E(BT) \cong E((B\mathbb{T})^n)$$

is the inclusion of the ring of invariants under the action of the Weyl group W . The coordinate gives an isomorphism

$$E(BT) \cong E((B\mathbb{T})^n) \cong E[\![x_1, \dots, x_n]\!],$$

with W acting as the permutation group Σ_n on the x_i 's. Thus we can define degree-zero Chern classes c_i in $E(BU(n))$ by the formula

$$\sum_{i=0}^n c_i z^{n-i} = \prod_{i=0}^n (z + x_i). \quad (2.4.2)$$

If F is the group law resulting from the coordinate x , then we call the c_i the “ F -Chern classes”.

Returning to the map

$$X \xrightarrow{h} Y,$$

as in (2.1), we can now define an umkehr map

$$E(X) \xrightarrow{h_F} E(Y),$$

using the degree-zero Thom isomorphism α . We write F to indicate the dependence on the coordinate.

DEFINITION 2.4.3. If X is any manifold, we denote by p^X the map

$$X \xrightarrow{p^X} *.$$

If E is an even periodic theory with group law F , then its F -genus is the element $p_F^X(1)$ of E .

2.5. The Riemann-Roch formula. The Riemann-Roch formula compares the umkehr homomorphisms h_F and h_G of two coordinates with formal group laws F and G , related by an isomorphism

$$\theta: F \rightarrow G.$$

The book of Dyer [Dye69] is a standard reference.

PROPOSITION 2.5.1. *If $h: X \rightarrow Y$ is a proper complex-oriented map of fiber dimension $2d$, then*

$$h_G(u) = h_F \left[u \cdot \prod_{j=1}^d \frac{x_j}{\theta(x_j)} \right], \quad (2.5.2)$$

where the x_i are the terms in the factorization

$$z^d + c_1 z^{d-1} + \dots + c_d = \prod_{j=1}^d (z + x_j)$$

of the total F -Chern class of the formal inverse of the normal bundle of h . □

REMARK 2.5.3. Changing the coordinate by a unit $u \in E$ multiplies the umkehr homomorphism by u^d ; by such a renormalization, we can always assume that θ is a *strict* isomorphism.

2.6. The fixed-point formula.

Notations for circle actions. Let \mathbb{T} denote the circle group \mathbb{R}/\mathbb{Z} . If X is a \mathbb{T} -space then we write

$$X_{\mathbb{T}} \stackrel{\text{def}}{=} E\mathbb{T} \times_{\mathbb{T}} X$$

for the Borel construction and $X^{\mathbb{T}}$ for the fixed-point set. Let $\mathbb{T}^* = \text{Hom}[\mathbb{T}, \mathbb{C}^\times]$ be the character group of \mathbb{T} ; we will also write $\hat{\mathbb{T}} = \mathbb{T}^* - \{1\}$ for the set of nontrivial irreducible representations. For $k \in \mathbb{T}^*$, let $\mathbb{C}(k)$ be the associated one-dimensional complex representation. There is then an associated complex line bundle $\mathbb{C}(k)_{\mathbb{T}}$ over $B\mathbb{T}$.

It is convenient to choose an isomorphism $\mathbb{T}^* \cong \mathbb{Z}$; this determines, in particular, an isomorphism $B\mathbb{T} \cong \mathbb{C}P^\infty$. For $k \in \mathbb{Z}$ we have $\mathbb{C}(k) = \mathbb{C}(1)^{\otimes k}$ and $\mathbb{C}(k)_{\mathbb{T}} = \mathbb{C}(1)_{\mathbb{T}}^{\otimes k}$. If $\hat{q} \in E(B\mathbb{T})$ is the Euler class of $\mathbb{C}(1)_{\mathbb{T}}$, then the Euler class of $\mathbb{C}(k)_{\mathbb{T}}$ is $[k](\hat{q})$.

Equivariant cohomology.

DEFINITION 2.6.1. Let (E, x, F) be a parametrized theory. A \mathbb{T} -equivariant cohomology theory $E_{\mathbb{T}}$ is an *extension* of (E, x, F) if

- (1) There is a natural transformation

$$E(X/\mathbb{T}) \rightarrow E_{\mathbb{T}}(X),$$

which is an isomorphism if \mathbb{T} acts freely on X . In particular the coefficient ring $E_{\mathbb{T}}(*)$ is an algebra over $E(*)$, and so it is 2-periodic.

- (2) There is a natural forgetful transformation

$$E_{\mathbb{T}}(X) \rightarrow E(X).$$

If X is a trivial \mathbb{T} -space then the composition

$$E(X) \rightarrow E_{\mathbb{T}}(X) \rightarrow E(X)$$

is the identity.

- (3) $E_{\mathbb{T}}$ has Thom classes and so Euler classes for complex \mathbb{T} -vector bundles, which are multiplicative and natural under pull-back. If V/X is such a bundle, then we write $e_{\mathbb{T}}(V) \in E_{\mathbb{T}}(X)$ for its (degree-zero) Euler class. These are compatible with the Thom isomorphism in E in the sense that, if the \mathbb{T} -action on V/X is trivial, then

$$e_{\mathbb{T}}(V) = e(V).$$

- (4) If L_1 and L_2 are complex \mathbb{T} -line bundles, then

$$e_{\mathbb{T}}(L_1 \otimes L_2) = e_{\mathbb{T}}(L_1) +_F e_{\mathbb{T}}(L_2).$$

DEFINITION 2.6.2. If $E_{\mathbb{T}}$ is equivariantly complex oriented as above, a homomorphism $E_{\mathbb{T}} \rightarrow \hat{E}_{\mathbb{T}}$ of multiplicative \mathbb{T} -equivariant cohomology theories is a *suitable* localization if

- (1) $\hat{E}_{\mathbb{T}}(*)$ is flat over $E_{\mathbb{T}}(*)$,
 1. When $k \neq 0$, $e_{\mathbb{T}}(\mathbb{C}(k))$ maps to a unit of $\hat{E}_{\mathbb{T}}(*)$, and
 (2) The fixed-point formula (2.6.4) holds for $\hat{E}_{\mathbb{T}}$.

In order to state the fixed-point formula, we need the following observation of [AS68].

LEMMA 2.6.3. *Let $E_{\mathbb{T}}$ be a suitable theory. Let S be a compact manifold with trivial \mathbb{T} -action, and let V be a complex \mathbb{T} -vector bundle over S . If the fixed-point bundle $V^{\mathbb{T}}$ is zero, then $e_{\mathbb{T}}(V)$ is a unit of $E_{\mathbb{T}}(S)$.*

PROOF. Recall [Seg68] that the natural map

$$\bigoplus_{k \in \hat{\mathbb{T}}} V(k) \otimes \mathbb{C}(k) \rightarrow V$$

is an isomorphism, where $V(k) \stackrel{\text{def}}{=} \text{Hom}[\mathbb{C}(k), V]$ is the evident vector bundle over S with trivial \mathbb{T} -action. By applying the ordinary splitting principle to $V(k)$, we are reduced to the case that $V = L \otimes \mathbb{C}(k)$, where L is a complex line bundle over S with trivial \mathbb{T} -action. If $E_{\mathbb{T}}$ is suitable then the Euler class of V is

$$e_{\mathbb{T}}(L \otimes \mathbb{C}(k)) = e(L) +_F e_{\mathbb{T}}(\mathbb{C}(k)).$$

Since S is a compact manifold, $e(L)$ is nilpotent in $E(S)$, so $e_{\mathbb{T}}(V)$ is a unit of $E_{\mathbb{T}}(S)$ because $e_{\mathbb{T}}(\mathbb{C}(k))$ is a unit of $E_{\mathbb{T}}(*)$. \square

Now suppose that M is a compact almost-complex manifold with a compatible \mathbb{T} -action. Let

$$j: S \rightarrow M$$

denote the inclusion of the fixed-point set; it is a complex-oriented equivariant map, with \mathbb{T} -equivariant normal bundle ν . The fixed-point formula which we require in Definition 2.6.2 is the equation

$$p_F^M(u) = p_F^S\left(\frac{j^*u}{e_{\mathbb{T}}(\nu)}\right). \quad (2.6.4)$$

By Lemma 2.6.3 $e_{\mathbb{T}}(\nu)$ is a unit of $E_{\mathbb{T}}(S)$, so this localization theorem is a corollary to the projection formula

$$j_F j^*(x) = x \cdot e_{\mathbb{T}}(\nu)$$

for the umkehr of the inclusion of the fixed-point set.

EXAMPLE 2.6.5. The Borel extension

$$E_{\text{Borel}}(X) \stackrel{\text{def}}{=} E(X_{\mathbb{T}})$$

of an even periodic ring theory has Thom classes $e_{\mathbb{T}}(V) = e(V_{\mathbb{T}})$ for complex \mathbb{T} -vector bundles, and the localization defined by inverting the multiplicative subset generated by $e_{\mathbb{T}}(\mathbb{C}(k)), k \neq 0$ will be suitable.

EXAMPLE 2.6.6. Let $K_{\mathbb{T}}$ denote the usual equivariant K -theory. Then $K_{\mathbb{T}} = \mathbb{Z}[q, q^{-1}]$, where q is the representation $\mathbb{C}(1)$, considered as a vector bundle over a point. The Euler class of a line bundle is $e_{\mathbb{T}}(L) = 1 - L$, so $\hat{q} = 1 - q$. The group law is multiplicative:

$$\mathbb{G}_m(x, y) = x + y - xy. \quad (2.6.7)$$

We have $[k](\hat{q}) = 1 - q^k$, and consequently

$$\hat{K}(X) \stackrel{\text{def}}{=} K_{\mathbb{T}}(X) \otimes_{K_{\mathbb{T}}} \mathbb{Z}((\hat{q}))$$

is suitable.

EXAMPLE 2.6.8. If

$$H_{\mathbb{T}}^*(X) \stackrel{\text{def}}{=} H^*(X_{\mathbb{T}}; \mathbb{Q}[v, v^{-1}])$$

is Borel cohomology with two-periodic rational coefficients, and $\hat{q} = e(\mathbb{C}(1)_{\mathbb{T}})$, then $H_{\mathbb{T}}(*) \cong \mathbb{Q}[[\hat{q}]]$, and $e(\mathbb{C}(k)_{\mathbb{T}}) = k\hat{q}$. The rational Tate cohomology

$$\hat{H}^*(X) \stackrel{\text{def}}{=} H_{\mathbb{T}}^*(X)[\hat{q}^{-1}]$$

is suitable.

EXAMPLE 2.6.9. More generally, two-periodic $\hat{K}(n)_{\mathbb{T}}$ (with n finite positive) is suitable: if $[p]_{K(n)}(X) = X^{p^n}$ and $k = k_0 p^s$ with $(k_0, p) = 1$ then

$$[k](\hat{q}) = [k_0](\hat{q}^{p^{ns}}) = k_0 \hat{q}^{p^{ns}} + \cdots \in \mathbb{F}_p((\hat{q}))$$

has invertible leading term. Integral lifts of $K(n)$ behave similarly; the Cohen ring [AMS98] of $\mathbb{F}_p((\hat{q}))$ defines a completion of the Borel-Tate localization.

These coefficient rings have natural topologies, which are relevant to the convergence of infinite products in Corollary 6.2.3.

3. Application to the free loop space

Let $\hat{E}_{\mathbb{T}}$ be a suitable localization (2.6.2) of an equivariantly complex oriented cohomology theory, let X be a compact complex-oriented manifold, and let $\mathcal{L}X$ be its free loop space. Since $\mathcal{L}X$ is not finite-dimensional, the existence of an umkehr homomorphism $p_F^{\mathcal{L}X}$ is not clear. However, \mathbb{T} acts on $\mathcal{L}X$ by rotations with fixed set X of constant loops, and Witten discovered that the fixed-point formula (2.6.4) for the F -genus $p_F^{\mathcal{L}X}(1)$ of $\mathcal{L}X$ continues to yield interesting formulae. In this section we review his calculation.

3.1. The normal bundle to the constant loops and its Euler class. One approximates the space $C^\infty(S^1, \mathbb{C})$ by the sub-vector space of Laurent polynomials

$$\mathbb{C}[\mathbb{T}^*] \cong \bigoplus_{k \in \mathbb{T}^*} \mathbb{C}(k) \hookrightarrow C^\infty(S^1, \mathbb{C}).$$

The tangent space of $\mathcal{L}X$ is $C^\infty(S^1, TX)$. If $p \in X$ is considered as a constant loop, then the tangent space to $\mathcal{L}X$ at p is the \mathbb{T} -space $T\mathcal{L}X_p \cong C^\infty(S^1, TX_p)$. It is a \mathbb{T} -bundle with a Laurent polynomial approximation

$$TX_p \otimes \mathbb{C}[\mathbb{T}^*]$$

Thus the normal bundle ν of the inclusion of X in $\mathcal{L}X$ has approximation

$$\nu \simeq \bigoplus_{k \in \hat{\mathbb{T}}} TX \otimes \mathbb{C}(k). \quad (3.1.1)$$

If

$$z^d + c_1 z^{d-1} + \dots + c_d = \prod_{j=1}^d (z + x_j)$$

is the formal factorization of the total F -Chern class of TX , then

$$e_{\mathbb{T}}(\nu) = \prod_{j=1}^d \prod_{k \neq 0} (x_j +_F [k]_F(\hat{q})), \quad (3.1.2)$$

where $\hat{q} = e_{\mathbb{T}}(\mathbb{C}(1))$.

3.2. The fixed point formula. Applying (2.6.4) to the inclusion

$$X \rightarrow \mathcal{L}X$$

yields the formula

$$p_F^{\mathcal{L}X}(1) = p_F^X \left[\prod_{j=1}^d \prod_{k \neq 0} \frac{1}{x_j +_F [k]_F(\hat{q})} \right]. \quad (3.2.1)$$

Equation (3.2.1) requires some interpretive legerdemain. For example, the leading coefficient of

$$\prod_{k \neq 0} (x +_F [k]_F(\hat{q}))$$

is the objectionable expression $\prod_{k \neq 0} (k\hat{q})$; but, as physicists say, this quantity is not directly ‘observable’. For this reason, we consider the *renormalized* formal product

$$\Theta_F(x; \hat{q}) \stackrel{\text{def}}{=} x \prod_{k \neq 0} \frac{(x +_F [k]_F(\hat{q}))}{[k]_F(\hat{q})}.$$

In section 6 below we provide a natural setting for such formal products.

The fixed point formula suggests that we define the equivariant F -genus of $\mathcal{L}X$ to be

$$\tilde{p}_F^{\mathcal{L}X}(1) \stackrel{\text{def}}{=} p_F^X \left[\prod_{j=1}^d \frac{x_j}{\Theta_F(x_j; \hat{q})} \right]. \quad (3.2.2)$$

3.3. Examples.

The additive group law. When F is the additive group law Θ_F becomes

$$\begin{aligned}\Theta_{\mathbb{G}_a}(x, \hat{q}) &= x \prod_{k \neq 0} \left(\frac{x}{k\hat{q}} + 1 \right) \\ &= x \prod_{k > 0} \left(1 - \frac{x^2}{k^2 \hat{q}^2} \right).\end{aligned}$$

This is the Weierstrass product for $\pi^{-1}\hat{q} \sin \hat{q}^{-1}\pi x$, so for the theory \hat{H} of (2.6.8), formula (3.2.2) gives

$$\tilde{p}_F^{\mathcal{L}X}(1) = (2\pi i/\hat{q})^d \left[\prod_{j=1}^d \frac{x_j/2}{\sinh(x_j/2)} \right] [X].$$

This is just the \hat{A} -genus of X , up to a normalization depending on the dimension of X . In [Ati85], Atiyah rewrites the formal product

$$x \prod_{k \neq 0} (x + k\hat{q})$$

as

$$x \left(\prod_{k > 0} k \right)^2 \prod_{k > 0} \left(\frac{x^2}{k^2} - \hat{q}^2 \right)$$

and invokes zeta-function renormalization [Den92] to replace $(\prod_{k > 0} k)^2$ with 2π , yielding

$$2\pi x \prod_{k > 0} \left(\frac{x^2}{k^2} - \hat{q}^2 \right);$$

specializing \hat{q} to i then gives the classical expression. From our point of view it's natural to think of the Chern class \hat{q} of $\mathbb{C}(1)$ as the holomorphic one-form $z^{-1}dz$ on the complex projective line, and thus to identify \hat{q} with its period $2\pi i$ with respect to the equator of \mathbb{CP}_1 as in §2.1 of [Del89]: the ‘Betti realization’ of the Tate motive $\mathbb{Z}(n)$ is $(2\pi i)^n \mathbb{Z} \subset \mathbb{C}$.

The multiplicative group law. In the case of the equivariant K -theory \hat{K} of example (2.6.6), the Euler class of a line bundle L is $e_{\mathbb{T}}(L) = 1 - L$. Writing q for the generator $\mathbb{C}(1)$ of \mathbb{T}^* , the Euler class of $\mathbb{C}(1)$ is

$$\hat{q} = e_{\mathbb{T}}(\mathbb{C}(1)) = 1 - q.$$

The multiplicative group law (2.6.7) gives

$$e_{\mathbb{T}}(L) +_{\mathbb{G}_m} [k](\hat{q}) = 1 - q^k L,$$

and so the formal product $\Theta_{\mathbb{G}_m}(x, \hat{q})$ becomes

$$\begin{aligned}\Theta_{\mathbb{G}_m}(x, \hat{q}) &= (1 - L) \prod_{k > 0} \frac{(1 - Lq^k)(1 - Lq^{-k})}{(1 - q^k)(1 - q^{-k})} \\ &= \left(\prod_{k > 0} L \right) (1 - L) \prod_{k > 0} \frac{(1 - Lq^k)(1 - L^{-1}q^k)}{(1 - q^k)^2}.\end{aligned}\tag{3.3.1}$$

Aside for the powers of L , this is essentially the product expansion for the Weierstrass σ function

$$\sigma(L, q) = (1 - L) \prod_{k > 0} \frac{(1 - q^k L)(1 - q^k L^{-1})}{(1 - q^k)^2} \in \mathbb{Z}[L, L^{-1}][[q]]$$

(see for example [MT91] or p. 412 of [Sil94]). The infinite factor is objectionable: in the product (3.2.2) defining the hypothetical \hat{K} -genus of $\mathcal{L}X$ this factor contributes an infinite power of $\Lambda^{\text{top}}TX$, but if $c_1 X = 0$

(e.g. if X is Calabi-Yau) and we are careful with the product, we can replace $\Theta_{\mathbb{G}_m}$ with θ . The resulting invariant is the Witten genus [Wit88, AHS98]. Segal [Seg88] replaces the formal product

$$\prod_{k \neq 0} (1 - q^k L)(1 - q^{-k} L)$$

which arises in the multiplicative case with

$$\left(\prod_{k > 0} q^{-k} L \right) \prod_{k > 0} (1 - q^k L)(1 - q^k L^{-1}).$$

He eliminates the infinite product of L 's by assuming $\Lambda^{\text{top}} TX$ trivial and he uses zeta-function renormalization to replace $q^{-\sum k}$ with $q^{-1/12}$.

4. A Riemann-Roch formula for the quotient of a formal group by a free subgroup

The starting point for this paper was the discovery that the formal products (3.2.1) and (3.2.2) which arise in applying the fixed point formula to study the F -genus of the free loop space are *precisely* the same as those obtained from the Riemann-Roch theorem for the quotient of F by a free cyclic subgroup. We explain this in section 4.2, after briefly reviewing finite quotients of formal groups, following [Lub67, And95].

4.1. The quotient of a formal group by finite subgroup. In this section, we assume that F is a formal group law over a complete local domain R of characteristic 0 and residue characteristic $p > 0$. If A is a complete local R -algebra, the group law F defines a new abelian group structure on the maximal ideal \mathfrak{m}_A of A . We will refer to $(\mathfrak{m}_A, +_F)$ as the group $F(A)$ of A -valued points of F .

If H is a *finite* subgroup of $F(R)$, then Lubin shows that there is a formal group law F/H over R , determined by the requirement that the power series

$$f_H(x) \stackrel{\text{def}}{=} \prod_{h \in H} (x +_F h) \in R[[x]] \quad (4.1.1)$$

is a homomorphism of group laws $F \xrightarrow{f_H} F/H$; in other words there is an equation

$$F/H(f_H(x), f_H(y)) = f_H(F(x, y)).$$

The main point is that the power series f_H is constructed so the kernel of f_H applied to $F(R)$ is the subgroup H .

The coefficient $f'_H(0) = \prod_{h \neq 0} h$ of the linear term of $f_H(x)$ is not a unit of R , and so f_H is an isomorphism of formal group laws only over $R[f'_H(0)^{-1}]$. Over this ring, we might as well replace f_H with the strict isomorphism

$$g_H(x) \stackrel{\text{def}}{=} x \prod_{h \neq 0} \frac{x +_F h}{h} = \frac{f_H(x)}{f'_H(0)},$$

and define G to be the formal group law

$$G(x, y) = g_H(F(g_H^{-1}(x), g_H^{-1}(y)))$$

over $R[f'_H(0)^{-1}]$; then F/H and G are related by the isomorphism

$$t(x) = f'_H(0)x.$$

If F is the group law and R the ring of coefficients of a parametrized theory, then the Riemann-Roch formula (2.5.2) for a compact complex-oriented manifold X is the equation

$$p_G^X(u) = p_F^X \left[u \prod_{j=1}^d \frac{x_j}{g_H(x_j)} \right] \quad (4.1.2)$$

over $R[f'_H(0)^{-1}]$.

4.2. The case of a free cyclic subgroup. Now suppose that (E, x, F) is a parametrized theory. Let R be the even periodic theory defined by the formula

$$R(X) \stackrel{\text{def}}{=} E(B\mathbb{T} \times X).$$

The projection $B\mathbb{T} \rightarrow *$ gives a natural transformation $E(X) \rightarrow R(X)$. In particular the coordinate $x \in E(P)$ gives a coordinate $x \in R(P)$. The group law \mathbf{F} is just the group law F , considered over the E -algebra R .

What extra structure is available over R ? A character $\lambda \in \mathbb{T}^*$ gives a map $B\mathbb{T} \rightarrow P$, and so an R -valued point $u(\lambda)$ of P_E . As λ varies through the group of characters, these points assemble into a homomorphism of groups

$$u: \mathbb{T}^* \rightarrow \mathbf{F}(R). \quad (4.2.1)$$

It is easy to see that this is the inclusion of a subgroup. As usual we write $\hat{q} = u(1)$ for the generator of \mathbb{T}^* using the isomorphism $\mathbb{T}^* \cong \mathbb{Z}$. We write (\hat{q}) for the subgroup $u(\mathbb{T}^*)$ generated by \hat{q} .

The analogue of Lubin's formula (4.1.1) is

$$f_{(\hat{q})}(x) = \prod_{k \in \mathbb{Z}} (x +_F [k](\hat{q})), \quad (4.2.2)$$

and the resulting Riemann-Roch formula

$$p_{\mathbf{F}/(\hat{q})}^X(u) = p_F^X \left[u \prod_{j=1}^d \prod_{0 \neq k} \frac{1}{x_j +_F [k]_F(\hat{q})} \right]$$

has right-hand side identical to the formal product (3.2.1) arising from the fixed-point formula.

Of course this does not avoid the problems of applying the fixed-point formula to the free loop space. From this point of view, the trouble is that the quotient object $\mathbf{F}/(\hat{q})$ isn't a formal group; for example the coefficient of x in the product (4.2.2) is $\prod_{k \neq 0} (k\hat{q})$. This particular problem is fixed by using

$$\begin{aligned} g_{(\hat{q})}(x) &= x \prod_{0 \neq k \in \mathbb{T}^*} \frac{x +_F [k]_F(\hat{q})}{[k]_F(\hat{q})} \\ &= \Theta_F(x, \hat{q}). \end{aligned}$$

The Riemann-Roch formula in this case is

$$p_G^X(u) = p_F^X \left[u \prod_{j=1}^d \frac{x_j}{\Theta_F(x_j, \hat{q})} \right] \quad (4.2.3)$$

with right-hand-side identical to the renormalized genus (3.2.2).

However, expanding such Weierstrass products as formal power series is still highly nontrivial, as the examples above have shown.

5. The structure of $\mathbf{F}/(\hat{q})$

Suppose now that F is a formal group law over a complete local ring E with residue field k of characteristic $p > 0$, and that \mathbf{F} is its pullback over the power series algebra $R = E[[\hat{q}]]$. The coordinate defines a homomorphism

$$\begin{aligned} u: \mathbb{Z} &\rightarrow \mathbf{F}(R) \\ u(n) &= [n]_{\mathbf{F}}(\hat{q}); \end{aligned} \quad (5.0.4)$$

but there is no reason to expect the cokernel of this homomorphism to be a formal group. However, $\mathbf{F}/(\hat{q})$ certainly makes sense as a group-valued functor of complete local R -algebras.

In section 5.1, we show by construction that the torsion subgroup of $\mathbf{F}/(\hat{q})$ has a natural approximation by a representable functor. We construct a formal group scheme $\text{Tate}(F)$ over R together with a natural transformation

$$\text{Tate}(F)_{\text{tors}} \rightarrow \mathbf{F}/(\hat{q})_{\text{tors}}$$

of group-valued functors, which is an isomorphism if \hat{q} has infinite order in F . The formal group scheme $\text{Tate}(F)_{\text{tors}}$ is our model for $\mathbf{F}/(\hat{q})$.

Because E has finite residue characteristic, we work p -locally: the formal group scheme $\text{Tate}(F)[p^\infty]$ is a p -divisible group in the sense of [Tat67, Dem72]. The p -torsion subgroup $G[p^\infty]$ of a formal group G of finite height is a *connected* p -divisible group, but $\text{Tate}(F)[p^\infty]$ is not; its maximal étale quotient is a constant height-one p -divisible group $\mathbb{Q}_p/\mathbb{Z}_p$, and its connected component is just the p -divisible group $F[p^\infty]$ of F . In other words, there is an extension

$$\mathbf{F}[p^\infty] \rightarrow \text{Tate}(F)[p^\infty] \rightarrow \mathbb{Q}_p/\mathbb{Z}_p$$

of p -divisible groups over R . We will see in (5.3.1) that it is in fact the *universal* example of such an extension, and it follows that if E_n is the ring which classifies lifts of a formal group G of height n over an algebraically closed field k [LT66], then $R = E_n[[\hat{q}]]$ represents the functor which classifies lifts of a p -divisible group of height $n+1$ with connected component $G[p^\infty]$. Thus \hat{q} may be viewed as a “Serre-Tate parameter” in the sense of [Kat81].

5.1. A model for the torsion subgroup of $F/(\hat{q})$. One difficulty in representing the quotient $\mathbf{F}/(\hat{q})$ over $E[[\hat{q}]]$ is that the subgroup (\hat{q}) does not act universally freely on \mathbf{F} : consider, for example, the specialization $\hat{q} = 0$. It turns out that, as long as one restricts to the torsion subgroup of $\mathbf{F}/(\hat{q})$, this is the only obstruction. By freeing up the action, we are able to construct a representable functor whose torsion subgroup coincides with that of $F/(\hat{q})$ whenever \hat{q} is of infinite order.

Following [KM85, §8.7], let $\text{Tate}(F)$ be the scheme over $R = E[[\hat{q}]]$ defined by the disjoint union

$$\text{Tate}(F) = \bigcup_{a \in \mathbb{Q} \cap [0,1)} F_R \times \{a\}.$$

If A is a complete local R -algebra, then

$$\text{Tate}(F)(A) = \{\text{pairs } (g, a) \text{ with } g \in F(A) \text{ and } a \in \mathbb{Q} \cap [0,1).\}$$

This has a group structure given by

$$(g, a) \cdot (h, b) = \begin{cases} (g +_F h, a + b) & \text{if } a + b < 1 \\ (g +_F h -_F \hat{q}, a + b - 1) & \text{if } a + b \geq 1. \end{cases}$$

By construction, $\text{Tate}(F)(A)$ is the quotient in the exact sequence

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbb{Z} & \rightarrow & F(A) \times \mathbb{Q} & \rightarrow & \text{Tate}(F)(A) \rightarrow 0 \\ & & n \mapsto & & ([n](\hat{q}), n) & & \\ & & & & (x, a) \mapsto & & (x -_F [\mathfrak{b}(a)](\hat{q}), \mathfrak{f}(a)), \end{array} \quad (5.1.1)$$

where $\mathfrak{b}(a)$ and $\mathfrak{f}(a)$ are the integral and fractional parts of the rational number a . Equivalently, $\text{Tate}(F)$ is the pushout of $\mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z}$ along the homomorphism u :

$$\begin{array}{ccccc} \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} \\ u \downarrow & & \downarrow & & \downarrow = \\ F_R & \longrightarrow & \text{Tate}(F) & \longrightarrow & \mathbb{Q}/\mathbb{Z}. \end{array}$$

Thus $\text{Tate}(F)$ is a kind of homotopy quotient of \mathbf{F} by \mathbb{Z} .

PROPOSITION 5.1.2. *Projection onto the first factor in the construction above defines a natural transformation*

$$\text{Tate}(F)_{\text{tors}} \rightarrow (\mathbf{F}/(\hat{q}))_{\text{tors}},$$

which is an isomorphism if \hat{q} is not in $F(A)_{\text{tors}}$.

PROOF. Let A be a complete local R algebra, and suppose that \hat{q} is not torsion in $F(A)$. We see from (5.1.1) that there is an isomorphism

$$\text{Tate}(F)(A) \cong \mathbf{F}/(\hat{q})(A) \times \mathbb{Q}$$

which is compatible with the projection and clearly induces an isomorphism

$$\text{Tate}(F)_{\text{tors}}(A) \cong (\mathbf{F}/(\hat{q}))_{\text{tors}}(A).$$

□

REMARK 5.1.3. The Proposition holds in any context in which \hat{q} is not torsion. Another source of examples is p -adic fields. Let $S = E[\frac{1}{p}](\langle \hat{q} \rangle)$. Let L be a complete nonarchimedean field with norm $\| - \|$. Given a continuous map $S \rightarrow L$, the group law \mathbf{F} defines a group structure on the set $\{v \in L \mid \|v\| < 1\}$; and we write $\mathbf{F}(L)$ for this group. The subgroup generated by \hat{q} is necessarily free, and the argument shows that there is an isomorphism of groups

$$\text{Tate}(F)_{\text{tors}}(L) \cong (\mathbf{F}/(\hat{q}))_{\text{tors}}(L).$$

REMARK 5.1.4. If the ring E is the ring of coefficients in an even periodic ring theory, then the preceding construction could be carried out with $R = E(B\mathbb{T})$ and the homomorphism u (4.2.1). The result is an extension

$$P_E \rightarrow \text{Tate}(P_E) \rightarrow (\mathbb{T}^* \otimes \mathbb{Q})/\mathbb{T}^*$$

of group schemes over R . Proposition 5.1.2 becomes an isomorphism

$$\text{Tate}(P_E)_{\text{tors}} \cong ((P_E)_R/\mathbb{T}^*)_{\text{tors}}.$$

5.2. Notation. If A is an abelian group, then $A_T = \text{spec } T^A$ is the resulting constant formal group scheme over T . The category of *test rings* is the category of Artin local E -algebras with residue field k .

DEFINITION 5.2.1. If G is a formal group scheme over a ring T , and A is an abelian group, let $\text{Hom}[A, G]$ be the group of homomorphisms

$$A_T \rightarrow G$$

of formal group schemes over T . Similarly, let $\text{Ext}[A, G]$ be the set of isomorphisms classes of extensions of formal group schemes

$$G \rightarrow X \rightarrow A_T.$$

If G is a formal group scheme over E , let $\underline{\text{Hom}}[A, G]$ and $\underline{\text{Ext}}[A, G]$ be the functors from test rings to groups such that

$$\underline{\text{Hom}}[A, G](T) \stackrel{\text{def}}{=} \text{Hom}[A_T, G_T]$$

and

$$\underline{\text{Ext}}[A, G](T) \stackrel{\text{def}}{=} \text{Ext}[A_T, G_T].$$

Now if G is a formal group over E , pulling back over G defines a natural point

$$\Delta: G \rightarrow G \times G = G_G$$

and hence a homomorphism

$$u: \mathbb{Z}_G \rightarrow G_G.$$

It is clear that this gives an isomorphism $G \cong \underline{\text{Hom}}[\mathbb{Z}, G]$.

Equivalently, if F is a formal group law over E , then $R = E[\langle \hat{q} \rangle]$ pro-represents the functor $\underline{\text{Hom}}[\mathbb{Z}, F]$ on the category of test rings, with universal example $u(n) = [n]_F(\hat{q})$ (5.0.4). Similarly, if E is the ring of coefficients of an even periodic ring theory, then $R = E(B\mathbb{T})$ pro-represents the functor $\underline{\text{Hom}}[\mathbb{T}^*, P_E]$, with the homomorphism u of (4.2.1) as the universal example.

5.3. Universal properties.

A universal extension. A continuous homomorphism of E -algebras from R to a test algebra T defines an extension

$$F_T \rightarrow (\mathrm{Tate}(F))_T \rightarrow (\mathbb{Q}/\mathbb{Z})_T$$

and hence an extension

$$F[p^\infty]_T \rightarrow (\mathrm{Tate}(F)[p^\infty])_T \rightarrow (\mathbb{Z}[\frac{1}{p}]/\mathbb{Z})_T.$$

of torsion subgroups.

LEMMA 5.3.1. *The ring R pro-represents the functor $\underline{\mathrm{Ext}}[\mathbb{Q}_p/\mathbb{Z}_p, F]$, with $\mathrm{Tate}(F)$ as the universal example.*

PROOF. In the exact sequence

$$\underline{\mathrm{Hom}}[\mathbb{Q}, F] \rightarrow \underline{\mathrm{Hom}}[\mathbb{Z}, F] \rightarrow \underline{\mathrm{Ext}}[\mathbb{Q}/\mathbb{Z}, F] \rightarrow \underline{\mathrm{Ext}}[\mathbb{Q}, F],$$

the first and last terms are zero because p acts nilpotently on F and as an isomorphism on \mathbb{Q} . \square

A universal p -divisible group. Any p -divisible group Γ over a field k is naturally an extension

$$\Gamma^0 \rightarrow \Gamma \rightarrow \Gamma_{\mathrm{et}} \tag{5.3.2}$$

of a connected group by an étale group. If the residue field k is algebraically closed, then the sequence (5.3.2) has a canonical splitting [Dem72, p. 34]. We will be interested in the case when Γ_{et} has height one, which is to say that it is isomorphic to the constant group scheme $(\mathbb{Q}_p/\mathbb{Z}_p)_k$.

Tate showed [Tat67] that the functor $G \mapsto G[p^\infty]$ is an equivalence between the categories of formal groups of finite height and connected p -divisible groups. Let's fix a one-dimensional formal group Γ^0 of height n over the algebraically closed field k and define Γ to be the product extension

$$\Gamma^0 \rightarrow \Gamma \rightarrow (\mathbb{T}^* \otimes \mathbb{Q}_p/\mathbb{Z}_p)_k.$$

DEFINITION 5.3.3. If G is a p -divisible group over k , and if T is a test ring, then a *lift* of G to R is a pair (F, δ) consisting of a p -divisible group F over R and an isomorphism

$$F_k \xrightarrow[\cong]{\delta} G$$

of p -divisible groups over k . An equivalence of lifts (F, δ) and (F', δ') is an isomorphism $f: F \rightarrow F'$ such that

$$\delta = \delta' f_k.$$

The set of isomorphism classes of lifts of G to R will be denoted $\mathrm{Lifts}_G(R)$. As R varies, $\mathrm{Lifts}_G(R)$ defines a functor from test rings to sets.

Lubin and Tate construct a formal power series algebra E_n over the Witt ring of k which pro-represents the functor $\mathrm{Lifts}_{\Gamma^0}$. There is an even periodic cohomology theory with E_n as ring of coefficients, and the universal lift F of Γ^0 as formal group.

THEOREM 5.3.4. *The ring $R = E_n(B\mathbb{T})$ pro-represents Lifts_Γ , with universal example $\mathrm{Tate}(F)[p^\infty]$.*

PROOF. Let T be a test ring. Suppose that (H, ϵ) is a lift of Γ . Then (H^0, ϵ^0) is a lift of Γ^0 . According to [LT66], there is a unique pair (f, a) consisting of a map $f: E_n \rightarrow T$ and an isomorphism

$$a: (F_T, \epsilon_T^{\mathrm{univ}}) \cong (H^0, \epsilon^0)$$

of lifts of Γ^0 . On the other hand, ϵ_{et} induces an isomorphism

$$H_{\mathrm{et}} = (H_{\mathrm{et}}(k))_T \cong (\Gamma_{\mathrm{et}}(k))_T = (\mathbb{T}^* \otimes \mathbb{Q}_p/\mathbb{Z}_p)_T.$$

Assembling these gives an extension

$$F_T \rightarrow H \rightarrow (\mathbb{T}^* \otimes \mathbb{Q}_p/\mathbb{Z}_p)_T$$

which defines an isomorphism

$$\text{Lifts}_R \cong \underline{\text{Ext}}[F, \mathbb{T}^* \otimes \mathbb{Q}_p/\mathbb{Z}_p] .$$

R pro-represents the right-hand side by Lemma 5.3.1. \square

REMARK 5.3.5. The analogous result for the ordinary multiplicative group \mathbb{G}_m is described in [KM85, §8.8]. Closely related examples occur in [AMS98], which are motivated by purely homotopy-theoretic questions about Mahowald's root invariant.

5.4. Examples. If $L \in A((q))$ is a unit, the formal product defining the Weierstrass function of §3.3 defines an element $\sigma(L, q) \in A((q))$. The functional equation $\sigma(qL, q) = (-L)^{-1}\sigma(L, q)$ then implies that the modified product

$$\sigma[L, r] \stackrel{\text{def}}{=} q^{-\flat(r)(\flat(r)+1)/2} (-L)^{\flat(r)} \sigma(L, q)$$

(where $\flat(r)$ is the integral part of $r \in \mathbb{Q}$) satisfies the identity $\sigma[qL, r+1] = \sigma[L, r]$ and can thus be regarded as a function from a localization of $\text{Tate}(\mathbb{G}_m)$ to the $\mathbb{Z}((q))$ -line. Since $\sigma(L, 0) = 1 - L$ we can think of the modified function as a deformation of the usual coordinate at the identity of the multiplicative group.

Similarly, if we regard \hat{q} as an element of the locally compact field \mathbb{C} , the modified sine function

$$s[x, r] \stackrel{\text{def}}{=} \pi^{-1} \hat{q} \sin \hat{q}^{-1}(\pi x - r)$$

satisfies the identity $s[x + \hat{q}, r+1] = s[x, r]$ and so can be interpreted as a function from $\text{Tate}(\mathbb{G}_a)(A)$ to A . It is also a deformation of the usual coordinate on the additive group, in that $s[x, 0] \rightarrow 0$ as $\hat{q} \rightarrow \infty$.

In general, however, there seems to be no reason to expect that a coordinate on F will extend to a coordinate on $\text{Tate}(F)$: our construction yields a group object, but not a group *law*. In the two examples above, we do have (something like) coordinates, which define interesting genera: ordinary cohomology leads to the \hat{A} -genus, suitably normalized, if $r = 0$; but if $r \neq 0 \in \mathbb{Q}/\mathbb{Z}$, Cauchy's theorem (applied to a small circle \mathbf{C} around the origin) yields

$$\frac{1}{2\pi i} \int_{\mathbf{C}} p_{\mathbb{G}_a}^X \left[\prod_{j=1}^{j=d} \frac{\hat{q}^{-1} \pi x_j z}{\sin \hat{q}^{-1} \pi (x_j z + r \hat{q})} \right] z^{-d-1} dz = \left(\frac{\pi}{\hat{q} \sin \pi r} \right)^d \chi(X) .$$

The K -theory genus extends similarly, but the resulting function is just $(-1)^d q^{-d\flat(r)(\flat(r)+1)/2}$ times the standard elliptic genus.

Any multiplicative cohomology theory E can be described as taking values in a category of sheaves over $\text{spec } E(*)$, and the Borel extension of such a theory takes values in sheaves over $\text{spec}(E(*)[[\hat{q}]])$. The construction in 5.1 of the Tate group as a disjoint union of copies of such affines implies that a theory E with formal group F has a natural extension to an equivariant theory taking values in a category of sheaves over the group object $\text{Tate}(F)$. Similarly, a suitable localization of the Borel extension defines an equivariant theory taking values in sheaves over a suitable localization of $\text{Tate}(F)$. This resembles (but is easier than) the constructions of [GKV, Gro94, RK99], for here we're only patching together Borel extensions.]

6. Prospectra and equivariant Thom complexes

Cohen, Jones, and Segal show that pro-objects in the category of spectra are the appropriate context in which to study Thom complexes, and so umkehr maps, for semi-infinite vector bundles. In this section we observe that the ideas of this paper fit very naturally into their framework.

6.1. Thom prospectra. If V and W are complex vector bundles over a space X , we extend the notation for Thom isomorphisms in 2.4.1 by writing

$$\alpha_W^V \stackrel{\text{def}}{=} \alpha^V \circ (\alpha^W)^{-1}: E(X^W) \rightarrow E(X^V) .$$

'Desuspending' by $V \oplus W$ then gives a homomorphism

$$\alpha_{-V}^{-W}: E(X^{-V}) \rightarrow E(X^{-W}) .$$

The inclusions of a filtered vector bundle

$$\mathcal{V} : 0 = V_0 \subset V_1 \subset \dots$$

define maps

$$i_{V_n} : X^{V_{n-1}} \rightarrow X^{V_n}$$

of Thom spectra, which desuspend to define a pro-object

$$X^{-\mathcal{V}} \stackrel{\text{def}}{=} \{\dots \rightarrow X^{-V_n} \xrightarrow{i^{V_n}} X^{-V_{n-1}} \rightarrow \dots\}$$

in the category of spectra. The E -cohomology of $X^{-\mathcal{V}}$ is the colimit

$$E(X^{-\mathcal{V}}) \stackrel{\text{def}}{=} \operatorname{colim}_n E(X^{-V_n}),$$

as in the appendix to [CJS95].

LEMMA 6.1.1. *On cohomology, the homomorphism induced by i^{V_n} is $e(V_n/V_{n-1})\alpha_{-V_{n-1}}^{-V_n}$.* \square

EXAMPLE 6.1.2. For any integer n , let

$$I_n = \bigoplus_{n \geq |k| > 0} \mathbb{C}(k);$$

thus $\mathcal{I} = \operatorname{colim}_n I_n$ is a filtered \mathbb{T} -vector bundle over a point. More generally, if V is a complex vector bundle over a space X , let

$$\mathcal{I}V \stackrel{\text{def}}{=} V \otimes \mathcal{I}$$

be the corresponding filtered \mathbb{T} -vector bundle. In this notation the Laurent approximation (3.1.1) to the formal normal bundle of the constant loops in $\mathcal{L}X$ is $\mathcal{I}TX$.

We write j_n for the map

$$j_n = i^{I_n V} : X^{-I_n V} \rightarrow X^{-I_{n-1} V}.$$

EXAMPLE 6.1.3. If V_n is the sum of n copies of $\mathbb{C}(1)$, considered as a \mathbb{T} -vector bundle over a point, then the Borel construction on \mathcal{V} is the Thom prospectrum for $\mathbb{C}P_\infty^\infty$ constructed in [CJS95].

6.2. Equivariant cohomology. Suppose now that $\hat{E}_\mathbb{T}$ is a suitable extension of an equivariantly oriented theory $E_\mathbb{T}$, and let X be a finite complex with trivial \mathbb{T} -action, as above. As n varies, the Thom isomorphisms

$$\alpha^{I_n V} : E_\mathbb{T}(X) \xrightarrow[\cong]{E_\mathbb{T}} (X^{-I_n V})$$

are not compatible with the maps j_n ; but this can be cured over $\hat{E}_\mathbb{T}$ by a suitable renormalization. By (2.6.3) the class

$$u_n(V) \stackrel{\text{def}}{=} e(V \otimes I_n) = \prod_{0 < |k| \leq n} e_\mathbb{T}(V \otimes \mathbb{C}(k)).$$

is a unit of $\hat{E}_\mathbb{T}(X)$, and the homomorphism

$$\omega_n(V) : \hat{E}_\mathbb{T}(X) \rightarrow \hat{E}_\mathbb{T}(X^{-I_n V})$$

defined by the formula $\omega_n(V) = u_n(V)\alpha^{-I_n V}$ is an isomorphism.

THEOREM 6.2.1. *The diagram*

$$\begin{array}{ccc} \hat{E}_\mathbb{T}(X) & \xlongequal{\quad} & \hat{E}_\mathbb{T}(X) \\ \omega_{n-1}(V) \downarrow & & \downarrow \omega_n(V) \\ \hat{E}_\mathbb{T}(X^{-I_{n-1} V}) & \xrightarrow{j_n} & \hat{E}_\mathbb{T}(X^{-I_n V}) \end{array}$$

commutes; in particular, the maps $\omega_n(V)$ assemble into a “Thom isomorphism”

$$\omega^V : \hat{E}_{\mathbb{T}}(X) \xrightarrow{\cong} \hat{E}_{\mathbb{T}}(X^{-\mathcal{I}V}) .$$

□

If V and W are two complex vector bundles over X , then (as with the usual Thom isomorphism) we define

$$\omega_W^V : \hat{E}_{\mathbb{T}}(X^{-\mathcal{I}W}) \xrightarrow{\cong} \hat{E}_{\mathbb{T}}(X^{-\mathcal{I}V}) .$$

COROLLARY 6.2.2. *If the vector bundle V has rank d , then the relative isomorphism*

$$\omega_d^V : \hat{E}_{\mathbb{T}}(X^{-\mathcal{I}d}) \rightarrow \hat{E}_{\mathbb{T}}(X^{-\mathcal{I}V})$$

is given by the formula

$$\omega_d^V = \left(\prod_{j=1}^d \prod_{k \neq 0} \frac{x_j +_F [k](\hat{q})}{[k](\hat{q})} \right) \alpha_{-\mathcal{I}d}^{-\mathcal{I}V} ,$$

where the x_j are the terms in the formal factorization (2.4.2) of the total F -Chern class of V . □

The terms in this product are well-defined at any finite stage, if not in the limit. If $E_{\mathbb{T}}$ is \hat{H} or \hat{K} (and $c_1(V) = 0$), then the infinite products converge and we have the

COROLLARY 6.2.3. *If the vector bundle V has rank d , then the diagram*

$$\begin{array}{ccc} \hat{E}_{\mathbb{T}}(X^d) & \xrightarrow{\alpha_d^{-\mathcal{I}d}} & \hat{E}_{\mathbb{T}}(X^{-\mathcal{I}d}) \\ \hat{\alpha}_d^V \downarrow & & \downarrow \omega_d^V \\ \hat{E}_{\mathbb{T}}(X^V) & \xrightarrow{\alpha_V^{-\mathcal{I}V}} & \hat{E}_{\mathbb{T}}(X^{-\mathcal{I}V}) . \end{array}$$

commutes; where

$$\hat{\alpha}_d^V \stackrel{def}{=} \frac{\Theta_F(V)}{c_d(V)} \alpha_d^V .$$

More precisely, the assertion is that these examples are naturally topologized, and that at any finite stage the diagram commutes modulo error terms which converge in this topology to zero as n grows. □

In other words, in a suitable theory with group law F the isomorphism ω_d^V is very much like a Thom isomorphism $\hat{\alpha}_d^V$ for a theory with F replaced by its extension $\text{Tate}(F)$. From this point of view, the fact that Witten’s formula for the α -genus of $\mathcal{L}X$ equals the $\hat{\alpha}$ -genus of X can be interpreted as saying that the inclusion $j : X \rightarrow \mathcal{L}X$ behaves as if there is a cohomological analogue

$$j_* : \hat{E}_{\mathbb{T}}(X^{-\mathcal{I}TX}) \rightarrow \hat{E}_{\mathbb{T}}(\mathcal{L}X)$$

of the Thom collapse map, with an associated umkehr $j_F = j_* \omega^{TX}$ satisfying the projection formula

$$j_F j^*(x) = x \cdot e_{\mathbb{T}}(\nu) .$$

Perhaps the intuition underlying the physicists’ interest in elliptic cohomology is that a reasonable equivariant theory applied to the geometrical object $X^{-\mathcal{I}TX}$ captures more information than that theory does, when applied directly to X . Thus the equivariant K -theory of this formal neighborhood of X is the Tate elliptic cohomology of X . This seems to be related to the recent construction (by Kontsevich and others) of new invariants for singular complex algebraic varieties, by considering them as varieties over the Laurent series field $\mathbb{C}((q))$.

6.3. Polarizations. In the preceding account, the role of the rational parameter in the construction of $\text{Tate}(F)$ is geometrically unmotivated, because we have ignored some issues connected with the polarization [CJS95] of the loop space.

Such a structure is an equivalence class of splittings of the tangent bundle of $\mathcal{L}X$ into a sum of positive- and negative-frequency components: if X is complex-oriented (*e.g.* symplectic, with a choice of compatible almost-complex structure), then the composition

$$\mathcal{L}X \rightarrow \mathcal{L}(BU) = U \times BU \rightarrow U/SO$$

of the map induced by the classifying map for the tangent bundle of X with the projection to a classifying space for such splittings defines a canonical polarization.

The universal cover of the free loop space. If X is simply-connected, the fundamental group of its free loop space will be isomorphic to $H_2(X, \mathbb{Z})$, which will be nontrivial in general. There is thus good reason to consider the simply-connected cover $\widetilde{\mathcal{L}X}$ of the loop space: this can be defined as the space of smooth maps of a two-disk to X , modulo the relation which identifies two maps if their restrictions to the boundary circle agree, and if furthermore their difference, regarded as an element of the deck-transformation group $\pi_2(X) = H_2(X, \mathbb{Z})$, is null-homotopic. The circle acts on $\widetilde{\mathcal{L}X}$ by rotating loops, as does the fundamental group of $\mathcal{L}X$, and in general the fixed-point set

$$\widetilde{\mathcal{L}X}^{\mathbb{T}} \cong H_2(X, \mathbb{Z}) \times X$$

will have many components. The choice of a basepoint defines a lift of the canonical polarization to a map

$$\widetilde{\mathcal{L}X} \rightarrow \mathbb{R} \times SU/SO$$

which restricts to a locally constant map

$$H_2(X, \mathbb{Z}) \times X \rightarrow \mathbb{R} \times SU/SO \rightarrow \mathbb{R}$$

defined by evaluating $\alpha \in H_2(X, \mathbb{Z})$ on the first Chern class of X . We think of the polarization as defining the zero-frequency modes in the Fourier decomposition of small loops near a fixed-point component, so that shifting by $\beta \in H_2(X, \mathbb{Z})$ gives an isomorphism

$$T\widetilde{\mathcal{L}X}_{x,\alpha}^{\mathbb{T}} \cong T\widetilde{\mathcal{L}X}_{x,\alpha+\beta}^{\mathbb{T}} \otimes \mathbb{C}(\langle c_1(V), \beta \rangle) .$$

\mathbb{T} -equivariant Picard groups and orientations. We can regard the cohomology of a space Y as a sheaf of rings over the zero-dimensional scheme

$$(\pi_0 Y)_{\mathbb{Z}} \stackrel{\text{def}}{=} \text{spec } H^0(Y, \mathbb{Z}) .$$

Similarly, the set of equivalence classes of \mathbb{T} -line bundles over a \mathbb{T} -space X is naturally isomorphic to $H^2(Y_{\mathbb{T}}, \mathbb{Z})$, which can be interpreted as a sheaf of groups over $(\pi_0 Y)_{\mathbb{Z}}$, the fiber above component Y_i being the constant group scheme

$$\text{coker } [H_{\mathbb{T}}^2(*, \mathbb{Z}) \rightarrow H_{\mathbb{T}}^2(Y_i)] ;$$

when the circle action on X is trivial, this is just a complicated way of indexing the summands of $H^2(Y, \mathbb{Z})$.

An orientation on a complex-oriented cohomology theory E with formal group F defines a natural homomorphism from the Picard group of line bundles over Y to the group of $E(Y)$ -valued points of F . This suggests that a complex orientation for the Tate extension of an equivariantly complex-oriented theory should be defined as a natural transformation from the Picard group of \mathbb{T} -line bundles over X to $\text{Tate}(F)(E(Y))$, both regarded as schemes over $(\pi_0 Y)_{\mathbb{Z}}$; by restriction such a transformation defines a map

$$\pi_0 Y \rightarrow \mathbb{Q}/\mathbb{Z} .$$

In other words, such a generalized orientation assigns to a component of Y and a \mathbb{T} -line bundle over it, a characteristic class in the cohomology of the component, together with an $r \in \mathbb{Q}$ depending on the component, which shifts by an integer when the bundle is twisted by a character.

When Y is $\widetilde{\mathcal{L}X}^{\mathbb{T}}$, the polarization defines a very natural map of this sort, which sends α to $\langle c_1(V), \alpha \rangle$. In more general situations, *e.g.* in Givental's work [Vo99] on the quantum cohomology of a symplectic

manifold (X, ω) the evaluation map $\alpha \mapsto \langle \omega, \alpha \rangle$ plays a similar role; but in this generalization, ω no longer needs to be an integral class.

7. Concluding remarks

The parts of this paper which deal principally with the fixed point formula on the free loop space are formulated in terms of a general equivariant cohomology theory $E_{\mathbb{T}}$, but those which relate to the quotient $\mathbf{F}/(\hat{q})$ use only the group law \mathbf{F} of the theory $R(X) = E(B\mathbb{T} \times X)$, and so essentially use the Borel theory E_{Borel} . We do this because in that case we can be more specific about our constructions. In good cases one can hope to do better.

Specifically, suppose that the multiplication homomorphism $\mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T}$ induces a map

$$E_{\mathbb{T}} \rightarrow E_{\mathbb{T} \times \mathbb{T}}$$

and that one has an isomorphism $E_{\mathbb{T} \times \mathbb{T}}(*) \cong E_{\mathbb{T}}(*) \otimes_E E_{\mathbb{T}}(*)$. The upshot is a group scheme $G = \text{spec } E_{\mathbb{T}}(*)$ over E , such that the formal group P_E associated to $E(P)$ is the completion of G . Indeed [GKV, Gro94], one expects in general that there is an abelian group scheme G which is the more fundamental object in \mathbb{T} -equivariant E -theory, with $E_{\mathbb{T}}(*)$ as structure sheaf.

In any case, as remarked in §5.2, over $G_G = G \times G$ there is a natural map

$$\mathbb{Z}_G \rightarrow G_G, \tag{7.0.1}$$

and the natural map $G_G \rightarrow \text{Hom}[\mathbb{Z}, G]$ is an isomorphism. One could consider the group $\text{Tate}(G)$ over G , fitting into a short exact sequence

$$G_G \rightarrow \text{Tate}(G) \rightarrow \mathbb{Q}/\mathbb{Z}.$$

The group $\text{Tate}(P_E)$ considered in the main text would then arise from $\text{Tate}(G)$ by completing in both copies of G .

This is the situation in K -theory. Over the ring $K_{\mathbb{T}} = \mathbb{Z}[q, q^{-1}]$ one has a homomorphism

$$\begin{aligned} u: \mathbb{Z} &\rightarrow \mathbb{G}_{mK_{\mathbb{T}}} \\ n &\mapsto q^n \end{aligned}$$

and this gives rise to a group $\text{Tate}(\mathbb{G}_m)$ over $K_{\mathbb{T}}$. Over the completion $K_{\mathbb{T}} \rightarrow K(B\mathbb{T}) = \mathbb{Z}[[\hat{q}]]$ one recovers the group $\text{Tate}(\mathbb{G}_m)$ considered in the main text. On the other hand, the group $\text{Tate}(\mathbb{G}_m)_{\text{tors}}$ becomes isomorphic to the torsion subgroup of the classical Tate curve $\text{Tate}(q)$ already over the suitable localization $\hat{K} = \mathbb{Z}((q))$ of (2.6.6), where u is the inclusion of a sub-groupscheme. See [KM85, §8.8] for details.

Interestingly enough, in the case of K -theory this *is* the solution to the problem of the infinite products. We have purposefully written the equivariant Euler class of $L \otimes \mathbb{C}(k)$ as $e(L) +_F [k]_F(\hat{q})$, as in any finite situation the formula

$$1 - L_1 L_2 = (1 - L_1) + (1 - L_2) - (1 - L_1)(1 - L_2)$$

makes it possible to calculate the Euler class of a vector bundle using the formal multiplicative group, i.e. the right-hand side. However, in order to calculate the infinite product $\Theta_{\mathbb{G}_m}(x, \hat{q})$ (3.3.1), one is *forced* to use the left-hand side. Thus from this point of view the renormalization is handled by knowing about the global multiplicative group, instead of merely its formal completion.

It seems reasonable to hope that the rich theory of \mathbb{T} -equivariant ring spectra will provide additional examples of such $E_{\mathbb{T}}$. If $E_{\mathbb{T}}$ is a complex-oriented \mathbb{T} -equivariant ring spectrum, then the cohomology $E_{\mathbb{T}}(\mathbb{CP})$ (where \mathbb{CP} is the space of lines in the ambient \mathbb{T} -universe) carries the structure of a \mathbb{T} -equivariant formal group law (in the sense of [CGK]). This is, among other things, a formal group law F and a homomorphism

$$v: \mathbb{T}^* \rightarrow F(E_{\mathbb{T}}(\mathbb{CP})).$$

In the situation we are considering, F is the completion of the group $G = \text{spec } E_{\mathbb{T}}$, and the homomorphism v is obtained from the homomorphism u by completion.

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THE UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

E-mail address: `mando@math.uiuc.edu`

THE JOHNS HOPKINS UNIVERSITY

E-mail address: `jack@math.jhu.edu`